SELF-SIMILAR SOLUTION OF THE PROBLEM OF THE HYDRODYNAMICS OF NONLINEAR VISCOUS FLUIDS

K. B. Pavlov

UDC 532.516

The laminar mixing of two parallel streams of a dilatant fluid is discussed. It is shown that the solution of this problem given in [1] contains a fundamental error resulting from ignoring the spatial localization of the mixing zone of the streams.

In the boundary-layer theory of non-Newtonian nonlinear viscous media there is a known solution of the problem of the mixing of two uniform laminar streams of a fluid whose rheological behavior is described by a power law

$$\sigma_{ij} = 2k\,(f_{\alpha\beta}f_{\alpha\beta})^{(n-1)/2}\,f_{ij}. \tag{1}$$

Here σ_{ij} is the stress tensor deviator, f_{ij} is the rate of strain tensor, and k and n are rheological constants of the fluid. According to accepted terminology, fluids with $n > 1$ are called dilatant, those with $n < 1$ pseudoplastic, and the case $n = 1$ corresponds to a viscous (Newtonian) fluid.

The numerical solution of the problem of the mixing of laminar streams of a power-law fluid is given in [1], but this treatment cannot be considered exhaustive since the characteristics of the structure of the mixing zone in dilatant fluids were undetected.

It is natural to expect a mixing zone for dilatant fluids which is strictly localized in space. This is related to the fact that shear perturbations are propagated with a finite velocity in dilatant fluids [2, 3]. Because of the motion of the medium in the direction of the longitudinal coordinate of the mixing zone, shear perturbations can propagate only a finite distance in the direction of the transverse coordinate. This causes the localization of the mixing zone in space. The analysis presented below confirms this conclusion, thus supplementing the results of [1].

Suppose a non-Newtonian fluid obeying the power law (1) moves in the direction of the x axis in the half space $x < 0$ with a velocity U_1 = const for $y > 0$ and U_2 = const for $y < 0$ (Fig. 1). For definiteness we assume $U_1 > U_2$. In the half space $x > 0$ the two streams are brought into contact and the flow in the mixing zone is described by the boundary layer equations of power-law fluids [4]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = a\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

Here u(x, y) and v(x, y) are the longitudinal and transverse components of the velocity of the fluid and $a \equiv k/\rho$, where ρ is its density. It was shown in [1] that the problem can be treated more generally by taking different values of k and ρ for the fluids in the two streams, but the spatial localization of the mixing zone for dilatant fluids can be established without this complication.

The function u(x, y) must obviously satisfy the conditions

$$u(x,\ \infty) = U_1,\ u(x,\ -\infty) = U_2. \tag{3}$$

N. É. Bauman Moscow Higher Technical School. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 36, No. 1, pp. 115-120, January, 1979. Original article submitted January 23, 1978.

0022-0841/79/3601-0083$07.50

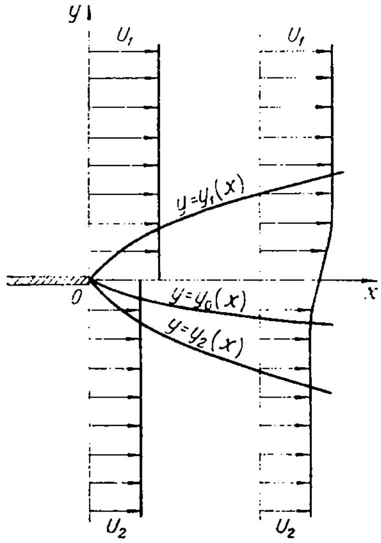

Fig. 1

By introducing the stream function

$$\psi(x,\ y) = (aU_1^{2n-1}x)^{1/(n+1)}\varphi(\eta), \tag{4}$$

$$u(x,\ y) = U_1\varphi'(\eta),\quad v(x,\ y) = (aU_1^{2n-1}x^{-n})^{1/(n+1)}[\eta\varphi'(\eta) - \varphi(\eta)]$$

(primes denote differentiation with respect to the self-similar variable)

$$\eta = y\left(\frac{U_1^{2-n}}{ax}\right)^{1/(n+1)} - \eta_0,\ \eta_0 \text{ — const}), \tag{5}$$

problem (2), (3) is reduced to the following:

$$n(n+1)(\varphi'')^{n-1}\varphi''' + \varphi\varphi'' = 0, \tag{6}$$

$$\varphi'(-\infty) = \Lambda \equiv U_2/U_1, \tag{7}$$

$$\varphi'(\infty) = 1. \tag{8}$$

Conditions (7) and (8) can be supplemented by the condition

$$\varphi(0) = 0, \tag{9}$$

corresponding to the fact that the stream function is zero on the boundary between the two streams in the half space x > 0. It follows from (5) that the equation of the boundary is given by the expression

$$y = y_0(x) \equiv \eta_0(aU_1^{n-2}x)^{1/(n+1)}, \tag{10}$$

which contains the unknown constant η_0 whose determination requires special consideration.

The solution of problem (6)-(9) determines the functions $\varphi(\eta),\ \varphi'(\eta)\ (-\infty<\eta<\infty)$ and, in particular, the value

$$\varphi'(0) = \lambda,\quad \Lambda < \lambda < 1. \tag{11}$$

If λ is assumed given, the solution of the original problem (6)-(9) for $0 \leq \eta < \infty$ is equivalent to the solution of a problem in which, in contrast with problem (6)-(9), boundary condition (11) is specified instead of boundary condition (7). We call (6), (8), (9), (11) a subsidiary problem.

For n ≠ 1 the left-hand side of Eq. (6) can be written as the product of two operators: $\{L[\varphi]\}\{M[\varphi]\} \equiv \left\{\frac{n(n+1)}{n-1}[(\varphi'')^{n-1}]' + \varphi\right\}\{\varphi''\}$. It seems plausible that the solution of the subsidiary problem should be obtained as a solution of the equation

$$L[\varphi] = 0 \tag{12}$$

with boundary conditions (8), (9), (11). This is actually so for $n < 1$, but is not true for $n > 1$. To prove this we start from physically obvious concepts of the properties of tangential shear stresses in the mixing zone of parallel streams. In accordance with these concepts we can assume that $\varphi''(\eta)$ is bounded and positive for any finite value of η ($0 \leq \eta < \infty$):

$$0 < \varphi''(\eta) < N < \infty, \quad N - \text{const.} \tag{13}$$

It follows from Eq. (13) and boundary conditions (8) and (11) that for any $Q > 0$ there exists an $\eta_Q > 0$ such that

$$\varphi(\eta > \eta_Q) > Q \tag{14}$$

and, in addition,

$$\varphi''(\infty) = 0. \tag{15}$$

By integrating Eq. (12) with $n > 1$ from 0 to η and using (14) it can be shown that for $\eta > \eta_Q$ the inequality

$$[\varphi''(\eta)]^{n-1} - [\varphi''(0)]^{n-1} = \frac{1-n}{n(1+n)} \int_0^\eta \varphi d\eta < \frac{1-n}{n(1+n)} \int_{\eta_Q}^\eta \varphi d\eta < \frac{Q(1-n)}{n(1+n)} (\eta - \eta_Q), \tag{16}$$

holds for all values of $\eta_Q < \eta < \infty$ for which Eq. (12) is valid. It follows from (13) and (15) that as $\eta \to \infty$ inequality (16) is meaningless. Thus, for $n > 1$ the solution of the subsidiary problem cannot be obtained as a solution of problem (12), (8), (9), (11).

We note that Eq. (6) is satisfied formally if

$$M[\varphi] = 0. \tag{17}$$

However, its solution

$$\varphi(\eta) = A + B\eta, \quad A, B\text{—const} \tag{18}$$

cannot simultaneously satisfy the three conditions (8), (9), (11). Therefore, the solution of the subsidiary problem for $n > 1$ cannot be obtained as a solution of problem (17), (8), (9), (11).

It can, however, be constructed as a generalized solution with different analytical descriptions in the domain $0 \leq \eta < \infty$:

$$\varphi(\eta) = \begin{cases} \varphi_0(\eta) & \text{for} \quad 0 \leqslant \eta \leqslant \eta_1 \\ \varphi_1(\eta) & \text{for} \quad \eta_1 \leqslant \eta < \infty, \end{cases} \tag{19}$$

where $\varphi_0(\eta)$ satisfies Eq. (12), and $\varphi_1(\eta)$ satisfies (17). It is essential that $L[\varphi_1] \neq 0$ for $\eta_1 \leq \eta < \infty$ and $M[\varphi_0] \neq 0$ for $0 \leq \eta \leq \eta_1$.

Before indicating the corresponding problems for determining $\varphi_0(\eta)$ and $\varphi_1(\eta)$, we note that it follows from the original Eq. (6) that φ, φ', and φ'' are continuous for all η including $\eta = \eta_1$, which is the quantity being sought.

The function $\varphi_1(\eta)$ defined by (18) must satisfy (8), and therefore in (18) the constant $B = B_1 \equiv 1$. The constant $A \equiv A_1$ must be found from the condition for the continuity of φ at $\eta = \eta_1$:

$$\varphi_1(\eta_1) = \varphi_0(\eta_1). \tag{20}$$

We now formulate the problem of determining φ_0.

It is clear that it must satisfy (9) and (11) for $\eta = 0$, and the following two conditions at $\eta = \eta_1$:

$$\varphi_0'(\eta_1) = 1, \quad \varphi_0''(\eta_1) = 0, \tag{21}$$

which follow from the continuity of $\varphi'(\eta)$ and $\varphi''(\eta)$ at $\eta = \eta_1$: $\varphi_0'(\eta_1) = \varphi_1'(\eta_1)$, $\varphi_0''(\eta_1) = \varphi_1''(\eta_1)$. The four conditions (9), (11), (21), and (12) define the problem. Its solution requires finding $\varphi_0(\eta)$ (19) and the boundary $\eta = \eta_1$.

By using (4) and (5), expressions can be written down for the longitudinal and transverse velocity components u(x, y) and v(x, y) for x > 0 and $y \geq y_1(x)$, where

$$y_1(x) = (\eta_0 + \eta_1)(aU_1^{n-2}x)^{1/(n+1)}, \quad n > 1, \tag{22}$$

determines the upper boundary of the mixing zone of the streams (Fig. 1). These expressions are

$$u(x,\ y) = U_1,\ v(x,\ y) = -A_1(aU_1^{2n-1}x^{-n})^{1/(n+1)}, \quad n > 1. \tag{23}$$

A similar procedure for n > 1 and negative values of η shows that a surface

$$y_2(x) = (\eta_0 + \eta_2)(aU_1^{n-2}x)^{1/(n+1)}, \quad n > 1. \tag{24}$$

must exist which bounds the mixing zone of the streams from below. For x > 0 and $y \leq y_2(x)$

$$u(x,\ y) = U_2,\ v(x,\ y) = -A_2(aU_1^{2n-1}x^{-n})^{1/(n+1)}, \quad n > 1, \tag{25}$$

where A_2 is the value of the constant A in solution (18) for $-\infty < \eta \leq \eta_2$.

Thus, for dilatant fluids (n > 1) the solution of the original problem (6)-(9) has different analytical forms for various values of η:

$$\varphi(\eta) = \begin{cases} \varphi_1(\eta) & \text{for} \quad \eta_1 \leqslant \eta < \infty \\ \varphi_0(\eta) & \text{for} \quad \eta_2 \leqslant \eta \leqslant \eta_1 \\ \varphi_2(\eta) & \text{for} \quad -\infty < \eta \leqslant \eta_2. \end{cases} \tag{26}$$

This is related to the spatial localization of the mixing zone of the two streams of dilatant fluids.

The spatial localization of the mixing zone for streams of dilatant fluids and the existence of the surfaces $y_1(x)$ (22) and $y_2(x)$ (24) alters the problem of determining the boundary of the two streams $y_0(x)$ (10) for n > 1. The Karman condition [5] can be used to determine $y_0(x)$. This corresponds to the vanishing of the transverse momentum in the mixing zone of the streams. As was shown in [1] the Karman condition for $n \leq 1$ can be written in the form

$$U_1 v(x,\ \infty) + U_2 v(x,\ -\infty) = 0. \tag{27}$$

It was erroneously stated in [1] that condition (27) holds for n > 1 also. Actually, because of the spatial localization of the mixing zone the condition for n > 1 must be written in the form

$$U_1 v[x,\ y_1(x)]\, y_1(x) + U_2 v[x,\ y_2(x)]\, y_2(x) = 0, \tag{28}$$

or, transforming to the self-similar variable η (5) and using (23)-(25), in the form

$$U_1 A_1(\eta_0 + \eta_1) + U_2 A_2(\eta_0 + \eta_2) = 0. \tag{29}$$

By using condition (29) the last unknown η_0 of the original problem can be determined, and the solution is complete.

It is impossible to write analytical expressions for $\varphi(\eta)$ (26) and the values of η_0, η_1, and η_2 for arbitrary n > 1 because there is no analytical expression for the solution of Eq. (12) except for the special case n = 2. It is appropriate to illustrate the general scheme for obtaining the expressions for $\varphi(\eta)$ (26) in this special case.

For n = 2 the problem of determining $\varphi(\eta)$ (26) has the form

$$\begin{gathered} \varphi_0''' + \omega^3\varphi_0 = 0, \quad \omega^3 = 1/6, \\ \varphi_0(0) = 0, \quad \varphi_0'(\eta_1) = 1, \quad \varphi_0'(\eta_2) = \Lambda, \\ \varphi_0''(\eta_1) = \varphi_0''(\eta_2) = 0. \end{gathered} \tag{30}$$

The solution of Eq. (30) which satisfies the first three conditions is

$$\varphi_0(\eta) = C_1 \exp[-\omega\eta] + e^{\omega\eta/2}\left(C_2 \cos\frac{\sqrt{3}}{2}\omega\eta + C_3 \sin\frac{\sqrt{3}}{2}\omega\eta\right),$$

$$C_1 = -C_2, \quad C_2 = \frac{\Delta_2}{\Delta}, \quad C_3 = \frac{\Delta_3}{\Delta},$$

$$\Delta \equiv \exp[3\omega(\eta_1 + \eta_2)/2] \sin\left[\frac{\sqrt{3}}{2}\omega(\eta_2 - \eta_1) - \exp[3\omega\eta_1/2] \cos \times \right.$$

$$\left. \times \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\omega\eta_1\right) + \exp[3\omega\eta_2/2] \cos\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\omega\eta_2\right)\right],$$

$$\Delta_2 \equiv \frac{1}{\omega}\left[\exp[\omega(\eta_1 + 3\eta_2/2)] \cos\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\omega\eta_2\right) - \right.$$

$$\left. - \Lambda \exp[\omega(\eta_2 + 3\eta_1/2)] \cos\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\omega\eta_1\right)\right],$$

$$\Delta_3 \equiv \frac{1}{\omega}\left[\Lambda \exp[-\omega\eta_2] - \exp[-\omega\eta_1] + \Lambda \exp[\omega(\eta_2 + 3\eta_1/2)] \sin \times \right.$$

$$\left. \times \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\omega\eta_1\right) - \exp[\omega(\eta_1 + 3\eta_2/2)] \sin\left(\frac{\pi}{6} - \frac{\sqrt{3}}{2}\omega\eta_2\right)\right].$$

Here ω is the real root of the equation $6\omega^3 = 1$.

Two transcendental equations for η_i $(i = 1, 2)$ can be obtained from the last two of equations (30):

$$C_1 \exp[-\omega\eta_i] + \exp[\omega\eta_i/2]\left[C_2 \sin\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\omega\eta_i\right) + C_3 \cos\left(\frac{\pi}{6} + \frac{\sqrt{3}}{2}\omega\eta_i\right)\right] = 0.$$

The expressions for $\varphi_i(\eta)$, $i = 1, 2$ (26) are found as solutions of the problem

$$\varphi_i'' = 0, \quad \varphi_i(\eta_i) = \varphi_0(\eta_i), \quad \varphi_1'(\infty) = 1, \quad \varphi_2'(-\infty) = \Lambda. \tag{31}$$

By solving (31) we have

$$\varphi_i(\eta) = A_i + B_i\eta, \quad i = 1, 2,$$

$$A_1 = \varphi_0(\eta_1) - \eta_1, \quad A_2 = \varphi_0(\eta_2) - \Lambda\eta_2,$$

$$B_1 = 1, \quad B_2 = \Lambda.$$

The last unknown η_0 is found from (29).

In conclusion, we note that the spatial localization of shear perturbations in dilatant fluids can also be observed in other boundary layer theory problems of media whose rheological behavior is described by (1).

LITERATURE CITED

1. C. W. Van Atta and T. T. Yeh, "Laminar free-shear layers in power-law fluids," AIChE J., 13, No. 6, 1203 (1967).
2. C. W. Van Atta, "An example of unsteady laminar mixing in power-law fluids," AIChE J., 12, No. 6, 1221 (1966).
3. L. K. Martinson and K. B. Pavlov, "Magnetohydrodynamics of non-Newtonian fluids," Mag. Gidrodin., No. 1, 59 (1975).
4. Z. P. Shul'man, Convective Heat and Mass Transfer of Rheologically Complex Fluids [in Russian], Énergiya, Moscow (1975).
5. Lu Ting, "On the mixing of two parallel streams," J. Mat. Phys., 38, No. 2, 153 (1959).